Axial Load

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Figure: 04-00CO
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Saint-Venant’s Principle

8. Axial Load
Saint-Venant’s Principle

Consider a prismatic member

- Material properties: $E, \nu$
- Physical dimensions: $L, A$
- Applied load: $P$
- Elongation: $\delta$
Elastic Deformation of an Axially Loaded Member

- 1-D Hooke’s law for linear elastic materials
  \[ \sigma = E \varepsilon \]
  \[ \Rightarrow \frac{P}{A} = E \frac{\delta}{L} \]

EA: axial rigidity
  \[ \Rightarrow \delta = \frac{PL}{EA} \text{ or } P = \frac{EA}{L} \delta \]

Example

Given \( E, A, \) find \( \delta_D \)
Example

Method 1: Elongation of each section

Section AB: $0 < x < L_a$

$\sum f_x = 0: -P_a + P(x) = 0$

$\therefore P(x) = P_a = P_2 + P_3 - P_1$

Section BC: $L_a < x < L_a + L_b$

$\sum f_x = 0: -P_a - P_a + P(x) = 0$

$\therefore P(x) = P_a = P_2 + P_3$

Section CD: $L_a + L_b < x < L$

$\sum f_x = 0: -P_a + P_a + P(x) = 0$

$\therefore P(x) = P_2 + P_3 - P_1 = P_1$

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Example

Method 1: Elongation of each section

Force-diagram:

$\delta_D = \delta_{AB} + \delta_{BC} + \delta_{CD}$

$= \frac{P L_a}{EA} + \frac{P L_b}{EA} + \frac{P L_c}{EA}$

$= \frac{(P_2 + P_3 - P_1)L_a}{EA} + \frac{(P_2 + P_3)L_b}{EA} + \frac{P_3L_c}{EA}$
Example

- Method 2: Superposition of each loading

\[ \delta_0^p = \delta_0 = -\frac{PL_a}{EA} \]

\[ \delta_0^p = \delta_0 = \frac{P_2(L_a + L_b)}{EA} \]

\[ \delta_0^p = \delta_0 = \frac{P_1(L_a + L_b + L_c)}{EA} \]

\[ \therefore \delta_0 = \delta_0^p + \delta_0^p + \delta_0^p = \frac{P_1L_a}{EA} + \frac{P_2(L_a + L_b)}{EA} + \frac{P_3(L_a + L_b + L_c)}{EA} \]

\[ = \frac{(P_1 + P_2 - P_1)L_a}{EA} + \frac{(P_2 + P_3)L_b}{EA} + \frac{P_3L_c}{EA} \]

Other Variations

\[ \delta_{total} = ? \]
**Indeterminate Structure**

- **Statically Determinate**: by drawing free-body diagram and solving equilibrium equations, all axial forces in the members and all reactions at the supports can be determined.

- **Statically Indeterminate**: the equilibrium equations are not sufficient for the calculation of axial forces and reactions.

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**Example**

Take free-body diagram of the whole structure:

\[
\sum f_x = 0: P - R_a - R_b = 0
\]

\[
\Rightarrow R_a + R_b = P \text{ (equilibrium equation)}
\]

\[
\Rightarrow \begin{cases} 
2 \text{ unknowns: } R_a, R_b \\
1 \text{ equation: equil. equation}
\end{cases}
\]

Need 1 more equation
(2 equ. 2 unknowns)
or
Reduce to 1 unknown
(1 equ. 1 unknown)
Indeterminate Structure

- Method 1: Flexibility method (find 1 more equation)

(i) Take $R_b$ away: statically determinate
For fixed end at $B$, no displacement

\[ \delta_B = \delta_B^P + \delta_B^R = 0 \]
(compatibility equation)

\[ \Rightarrow \frac{Pa}{EA} - \frac{R_bL}{EA} = 0 \]
\[ \Rightarrow R_b = \frac{Pa}{L} \]
From equi. eqn.
\[ R_a = P - R_b = \frac{Pb}{L} \]

(ii) Put $R_b$ back to restore original state

\[ \delta_B^R = \frac{R_bL}{EA} \]

- Method 2: Stiffness method (reduce to 1 equation)

Assume displacement at $C$ is $\delta_C$,
try to relate $R_a$ and $R_b$ to $\delta_C$

\[ \delta_C = \delta_C = \frac{R_a}{EA} \rightarrow \]
\[ \delta_C = \delta_C = \delta_C = \frac{R_b}{EA} \rightarrow \]

\[ \Rightarrow EA \frac{\delta_C}{a} + EA \frac{\delta_C}{b} = P \]
\[ \Rightarrow \delta_C = \frac{Pab}{EA(a + b)} = \frac{Pab}{EAL} \]
\[ \therefore R_a = \frac{Pb}{L}, R_b = \frac{Pa}{L} \]
Thermal Stress

Thermal expansion coefficient

\[ \varepsilon_x = \frac{\Delta a}{a} \Delta T \]
\[ \Rightarrow \varepsilon_x = \alpha_x \Delta T \]
\[ \varepsilon_y = \frac{\Delta b}{b} \Delta T \]
\[ \Rightarrow \varepsilon_y = \alpha_y \Delta T \]
\[ \varepsilon_z = \frac{\Delta c}{c} \Delta T \]
\[ \Rightarrow \varepsilon_z = \alpha_z \Delta T \]

For isotropic materials (also for cubic materials): \( \alpha_x = \alpha_y = \alpha_z = \alpha \)

Thermal Strain: \( \varepsilon_T = \alpha \Delta T \)

\( \alpha \): thermal expansion coefficient, unit: \( ^\circ\text{C}, ^\circ\text{F}, 1/\text{K} \)

Thermal Stress

For 1-D axial member (determinate structure)

\[ \delta_T = L \varepsilon_T = L \alpha \Delta T \]

No thermal stress: free expansion, no constrain

\[ \sigma_T = 0 \]
Thermal Stress

For constrained member (indeterminate structure)

\[ T_0 \to T_0 + \Delta T \]

No thermal strain

\[ \varepsilon_T = \frac{\Delta}{L} = 0; \quad \therefore \delta = 0 \]

Thermal stress?

\[ \sigma_T = ? \]

\[ R = \sum f_x = 0 \]

Equilibrium eqn:

\[ R - R = 0 \]

\[ \therefore 0 = 0 \]

Indeterminate structure

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Thermal Stress

Use flexibility method

(i) Take \( R \) away: statically determinate, relaxed structure or free expansion

\[ T_0 + \Delta T \]

\[ \delta_T = L \alpha \Delta T \]

(ii) Put \( R \) back to restore original state

\[ T_0 + \Delta T \]

\[ \delta_R = -\frac{RL}{EA} \]

For fixed end, no displacement

\[ \delta = \delta_T + \delta_R = 0 \]

(compatibility equation)

\[ L \alpha \Delta T - \frac{RL}{EA} = 0 \]

\[ \Rightarrow R = EA \alpha \Delta T \]

Thermal Stress:

\[ \sigma_T = \frac{R}{A} = E \alpha \Delta T \]

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