Abstract:
Because of decision makers having limited attention spans and information processing capabilities, as well as some decision alternatives being incomparable, decision makers may develop incomplete preference relations in which some elements cannot be provided. The method proposed by Herrera et al. devises a consistent preference relation that is restricted by the set of $p_{12}, p_{23}, \ldots, p_{n-1n}$, $p_{12}, p_{23}, \ldots, p_{n-1n}$. Therefore, for convenience and flexibility, the following uses the incomplete fuzzy preference relation with the least judgments (that is, $n - 1$ judgments) to develop a simple and practical method for constructing a consistent complete fuzzy preference relation in which experts can compare any row, column or diagonal. The proposed method is more convenient and flexible than that of Herrera et al.

Keywords:
Incomplete fuzzy preference relations; consistency; decision making; fusion

1. Introduction

Most decision processes are known to be based on preference relations [1,3,10,15], in that processes are linked to some degree of preference of one alternative over another. A well-known method of decision problems is the Analytic Hierarchy Process (AHP) developed by Saaty [8]. AHP separates a complex decision issue into elemental problems to establish a hierarchical model. When the decision problem is divided into smaller constituent parts in a hierarchy, pairwise comparisons of the relative importance of elements are performed in each level of the hierarchy to establish a set of weights or priorities. Although AHP is widely utilized in diverse fields [2,5,6,7,10,16], increasing the hierarchies of criteria or alternatives is always associated with inconsistency. Additionally, each of these preference relations necessitates the completion of all $\frac{n(n-1)}{2}$ judgements throughout its top triangular portion. However, it is sometimes difficult to yield such a complete preference relation, particularly for high order preference relations, because: (1) a decision should be made under time pressure and lack of data, and (2) the decision makers have limited expertise relating to the problem domain.

Due not only to decision makers having limited attention and information processing capabilities, but also to certain decision alternatives being incomparable, decision makers may develop incomplete preference relations in which some elements cannot be provided. Consequently, it is worth paying more attention to this issue. To resolve this dilemma, Herrera-Viedma et al. [11] developed a consistent fuzzy preference relation for facilitating decision making, enhancing the effectiveness and accuracy of decision making. However, in this method the construction of a consistent preference relation is restricted by the set of $p_{12}, p_{23}, \ldots, p_{n-1n}$.

For convenience and flexibility, this study developed a method, based on the consistent incomplete fuzzy preference relation involving least judgments, for constructing a consistent complete fuzzy preference relation using the multiplicative transitivity property. The increasing complexity and uncertainty of the socio-economic environment is assumed to reduce the likelihood that single decision makers will consider all aspects of a decision making problem; consequently, numerous real world decision making processes occur in multi-person settings, so this investigation designs a multi-person decision making approach based on the constructed consistent complete fuzzy preference relations, which fuses individual preferences into collective ones and aggregates the overall information regarding each decision alternative to rank alternatives and select the optimal one(s). Finally, an illustrative example is presented to verify the developed approach.

2. The use of the preference relations in decision making

Preference relation is the most common representation
of information used in decision making problems because it is a useful tool for modelling decision processes, particularly when aggregating the preferences of experts to produce group preferences [4,8,9]. Herrera et al. [11] proposed consistent fuzzy preference relations in accordance with two preference relations, namely multiplicative preference relation and fuzzy preference relations [12,13,14].

(1) Multiplicative preference relation. The preferences of experts regarding a set of alternatives \( \mathcal{X} \) can be denoted via a preference relation matrix \( A \subseteq \mathcal{X} \times \mathcal{X} \), \( A = (a_{ij}) \), where \( a_{ij} \in [\frac{1}{9}, 9] \), denotes the ratio of the preference degree of alternative \( x_i \) over \( x_j \). Since \( a_{ij} = 1 \) indicates indifference between \( x_i \) and \( x_j \), \( a_{ij} = 9 \) indicates that \( x_i \) is strongly preferred to \( x_j \). \( A \) is assumed to be a multiplicative reciprocal, namely

\[
a_{ij} \cdot a_{ji} = 1 \tag{1}
\]

Definition 1. A reciprocal multiplicative preference relation \( A = (a_{ij}) \) is consistent if

\[
a_{ij} \cdot a_{jk} = a_{ik} \quad \forall i, j, k = 1, \ldots, n. \tag{2}
\]

(2) Fuzzy preference relation. Expert preferences regarding a set of alternatives \( \mathcal{X} \) are denoted via a preference relation matrix \( P \subseteq \mathcal{X} \times \mathcal{X} \), with membership function: \( \mu_p : \mathcal{X} \times \mathcal{X} \rightarrow [0,1] \), where

\[
\mu_p(x_i, x_j) = p_{ij} \text{ indicates the ratio of the preference intensity of alternative } x_i \text{ to that of } x_j. \text{ If } p_{ij} = \frac{1}{2} \text{ implies indifference between } x_i \text{ and } x_j (x_i \sim x_j).
\]

\[
p_{ij} = 1 \text{ indicates } x_i \text{ is absolutely preferred to } x_j, \quad p_{ij} = 0 \text{ indicates } x_j \text{ is absolutely preferred to } x_i, \quad \text{and } p_{ij} > \frac{1}{2} \text{ indicates that } x_i \text{ is preferred to } x_j (x_i > x_j).
\]

\( P \) is assumed to be an additive reciprocal, given by

\[
p_{ij} + p_{ji} = 1 \tag{3}
\]

Proposition 1. Assume the existence of a set of alternatives, \( \mathcal{X} = \{x_1, x_2, \ldots, x_n\} \), which is associated with a reciprocal multiplicative preference relation \( A = (a_{ij}) \), with \( a_{ij} \in [\frac{1}{9}, 9] \). The corresponding reciprocal fuzzy preference relation, \( P = (p_{ij}) \) with \( p_{ij} \in [0,1] \), associated with \( A \) is then given as follows:

\[
p_{ij} = g(a_{ij}) = \frac{1}{2} \cdot (1 + \log_a a_{ij}) \tag{4}
\]

Proposition 2. Reciprocal additive fuzzy preference relations

\[
p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall i, j, k \tag{5}
\]

\[
p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall i < j < k \tag{6}
\]

\[
p_{(i-1)j} + p_{(i-1)k} + \cdots + p_{(j-1)j} + p_{jk} = \frac{3n}{2} \quad \forall i < j \tag{7}
\]

Notably, according to Proposition 2, constructing consistent fuzzy preference relations only requires \( n-1 \) \( \{p_{12}, p_{23}, \ldots, p_{n-1n}\} \) judgments; the other incomplete elements can be done through additive transitivity. If the preference matrix contains values outside the interval \([0,1]\), namely within the interval \([-a,1+a]\), a linear transform is required to preserve the reciprocity and additive transitivity, that is

\[
f(x) = \frac{x + a}{1 + 2a} \tag{8}
\]

For more details can refer to the study of Herrera et al. [11].

The method of Herrera et al. [11], can improve the performance of decision processes by reducing comparison times, but the construction of a consistent preference relation is restricted by the set of \( n-1 \) values \( \{p_{12}, p_{23}, \ldots, p_{n-1n}\} \). Therefore, for more convenience and flexibility, the following develops a simple and practical method for constructing a consistent complete fuzzy preference relation, based on the incomplete fuzzy preference relation with the least judgments (i.e. \( n-1 \) judgments), as follows:

Step 1. For a multi-person decision-making problem, let \( D = \{d_1, d_2, \ldots, d_m\} \) denote the set of decision makers, while \( \mathcal{X} = \{x_1, x_2, \ldots, x_n\} \) represents a discrete set of alternatives. The decision maker \( d_k \in D \) compares each pair of alternatives using the discrete term set \( A_k \), where \( A_k = (a_{ij}^k)_{n \times n} \), \( a_{ij}^k \in [\frac{1}{9}, 9] \). An incomplete preference relation \( A_k = (a_{ij}^k)_{n \times n} \) is then constructed with only \( n-1 \) judgments, where experts can choose any row, column or diagonal to compare.

Step 2. Utilize the known elements in \( A_k \) and Eq.(2) to
determine all the unknown elements in $A_k$ and thus derive a consistent and complete preference relation $A_k = (a^k_{ij})_{mn}$. 

Step 3. Utilize Eqs.(4) and (8) to translate the complete preference relation $A_k = (a^k_{ij})_{mn}$ into the complete fuzzy preference relation $\hat{A}_k = (\hat{a}^k_{ij})_{mn}$. 

Step 4. Utilize the averaging operator $\hat{a} = \frac{1}{n}(a_k \oplus a_l \oplus a_m \oplus \ldots \oplus a_n)$ for all $i, j$ (9) to fuse all the consistent complete fuzzy preference relations $\hat{A}_k = (\hat{a}^k_{ij})_{mn}$ $(k=1,2,...,m)$ into a collective complete fuzzy preference relation $\hat{A} = (\hat{a}^k_{ij})_{mn}$. 

Step 5. Utilize the averaging operator to fuse all the fuzzy preference degrees $\hat{a}_{ij}$ $(j=1,2,...,n)$ in the $i$th line of the $\hat{A}$, and obtain the average $\hat{a}_i$ of the $i$th alternative over all the other alternatives. 

Step 6. Rank all the alternatives $x_i$ $(i=1,2,...,n)$ and select the optimal one(s) according to the values of $\hat{a}_i$ $(i=1,2,...,n)$. 

Step 7. End.

3. Numerical example

This section presents a decision-making problem involving the evaluation of five candidates $x_i$ $(i=1,2,...,5)$. The problem involves four decision makers $d_k$ $(k=1,2,...,4)$ who compare these five alternatives using the discrete term set $A_k$, where $A_k = (a^k_{ij})_{mn}$, $a^k_{ij} \in [\frac{1}{9},9]$, and provide the following judgments:

$d_1: a^1_{12} = \frac{1}{5}$, $a^1_{13} = 2$, $a^1_{14} = 3$, $a^1_{15} = \frac{1}{2}$

$d_2: a^2_{31} = 5$, $a^2_{32} = 3$, $a^2_{34} = \frac{1}{5}$, $a^2_{35} = \frac{1}{3}$

$d_3: a^3_{21} = 5$, $a^3_{31} = \frac{1}{2}$, $a^3_{41} = 3$, $a^3_{51} = \frac{1}{3}$

$d_4: a^4_{12} = 5$, $a^4_{23} = 3$, $a^4_{34} = \frac{1}{5}$, $a^4_{45} = \frac{1}{3}$

Obtaining the best alternative(s) involves the following steps:

Step 1. Use Eq.(1) and the above information provided by $d_k$ $(k=1,2,...,4)$ to derive the incomplete preference relations $A_k = (a^k_{ij})_{mn}$, respectively, where “-” represents the unknown variable.

$A_1 = (a^1_{ij})_{5\times5} = \begin{bmatrix} 1 & \frac{1}{5} & 2 & \frac{1}{2} & \frac{1}{2} \\ 5 & 1 & - & - & - \\ - & 1 & \frac{1}{5} & - & - \\ \frac{1}{3} & - & - & 1 & - \\ 2 & - & - & 1 & - \end{bmatrix}$

$A_2 = (a^2_{ij})_{5\times5} = \begin{bmatrix} 1 & \frac{1}{5} & 3 & \frac{1}{5} & 1 \\ 5 & 3 & 1 & \frac{1}{5} & \frac{1}{3} \\ - & - & 5 & 1 & - \\ - & - & - & 3 & - \\ - & - & 3 & - & 1 \end{bmatrix}$

$A_3 = (a^3_{ij})_{5\times5} = \begin{bmatrix} 1 & \frac{1}{5} & 3 & \frac{1}{5} & 5 \\ 5 & 1 & - & - & - \\ \frac{1}{3} & - & - & 1 & - \\ 3 & - & - & 1 & - \\ \frac{1}{5} & - & - & 1 & - \end{bmatrix}$

$A_4 = (a^4_{ij})_{5\times5} = \begin{bmatrix} 1 & 5 & - & - & - \\ \frac{1}{5} & 1 & 3 & - & - \\ - & - & 1 & \frac{1}{4} & - \\ - & 4 & 1 & \frac{1}{3} & - \\ - & - & - & 3 & 1 \end{bmatrix}$

Step 2. Utilize the known elements in $A_k$ $(k=1,2,3,4)$ and Eq. (2) to determine all the unknown elements in $A_k$ $(k=1,2,3,4)$:
\[
A_1 = (a_{ij}^1)_{5 \times 5} = \begin{bmatrix}
1 & \frac{1}{3} & 2 & 3 & \frac{1}{2} \\
5 & 1 & 10 & 15 & \frac{1}{4} \\
\frac{1}{3} & \frac{1}{5} & \frac{2}{3} & 1 & \frac{1}{6} \\
2 & 4 & 4 & 6 & 1
\end{bmatrix}
\]
\[
a^1_{45} = a^1_{41} \cdot a^1_{15} = \frac{a^1_{41}}{a^1_{12}} \quad a^1_{54} = \frac{1}{a^1_{45}}
\]
\[
A_2 = (a_{ij}^2)_{5 \times 5} = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{5} & \frac{1}{15} \\
\frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{15} & \frac{1}{9} \\
5 & 3 & 1 & \frac{1}{5} & \frac{1}{3} \\
25 & 15 & 5 & 1 & \frac{5}{3} \\
15 & 9 & 3 & \frac{3}{5} & 1
\end{bmatrix}
\]
\[
a^2_{12} = a^2_{13} \cdot a^2_{32} = \frac{a^2_{13}}{a^2_{11}} \quad a^2_{21} = \frac{1}{a^2_{12}}
\]
\[
A_3 = (a_{ij}^3)_{5 \times 5} = \begin{bmatrix}
1 & \frac{1}{3} & 3 & \frac{1}{5} & 5 \\
5 & 1 & 15 & \frac{5}{3} & 25 \\
\frac{1}{3} & \frac{1}{5} & 9 & 1 & 15 \\
\frac{1}{5} & \frac{1}{5} & \frac{3}{5} & \frac{1}{15} & 1
\end{bmatrix}
\]
\[
a^3_{23} = a^3_{21} \cdot a^3_{13} = \frac{a^3_{21}}{a^3_{12}} \quad a^3_{32} = \frac{1}{a^3_{23}}
\]
\[
A_4 = (a_{ij}^4)_{5 \times 5} = \begin{bmatrix}
1 & 5 & 15 & \frac{15}{4} & \frac{5}{4} \\
\frac{1}{5} & 1 & 3 & \frac{3}{4} & \frac{1}{4} \\
\frac{1}{15} & \frac{1}{3} & 1 & \frac{1}{4} & \frac{1}{12} \\
\frac{4}{5} & \frac{4}{3} & 4 & 1 & \frac{1}{3} \\
\frac{4}{5} & 4 & 12 & 3 & 1
\end{bmatrix}
\]
\[
a^4_{24} = a^4_{21} \cdot a^4_{14} = \frac{a^4_{21}}{a^4_{12}}
\]
\[
a^4_{25} = a^4_{21} \cdot a^4_{15} = \frac{a^4_{21}}{a^4_{12}}
\]
\[
a^4_{32} = a^4_{31} \cdot a^4_{12} = \frac{a^4_{31}}{a^4_{13}} \quad a^4_{34} = \frac{1}{a^4_{32}}
\]
\[
a^4_{34} = a^4_{31} \cdot a^4_{14} = \frac{a^4_{31}}{a^4_{13}} \quad a^4_{35} = \frac{1}{a^4_{34}}
\]
\[
a^4_{45} = a^4_{41} \cdot a^4_{15} = \frac{a^4_{41}}{a^4_{14}} \quad a^4_{54} = \frac{1}{a^4_{45}}
\]
$$a_{24}^4 = a_{23}^4 \cdot a_{34}^4 \quad a_{42}^4 = \frac{1}{a_{24}^4}$$

$$a_{25}^4 = a_{24}^4 \cdot a_{45}^4 \quad a_{52}^4 = \frac{1}{a_{25}^4}$$

$$a_{35}^4 = a_{34}^4 \cdot a_{45}^4 \quad a_{53}^4 = \frac{1}{a_{35}^4}$$

Step 3. Use Eqs. (4) and (8) to obtain the corresponding consistent complete fuzzy preference relations

\[ \hat{A}_k = (\hat{a}_{ij})_{\infty} : \]

\[
\begin{bmatrix}
0.5 & 0.2028 & 0.6280 & 0.7028 & 0.3720 \\
0.7972 & 0.5 & 0.9251 & 1 & 0.6692 \\
0.3720 & 0.0749 & 0.5 & 0.5749 & 0.2440 \\
0.2972 & 0 & 0.4251 & 0.5 & 0.1692 \\
0.6280 & 0.3308 & 0.7560 & 0.8308 & 0.5
\end{bmatrix}
\]

\[ \hat{A}_1 = (\hat{a}_{ij})_{k=5} = \]

\[
\begin{bmatrix}
0.5 & 0.4207 & 0.2500 & 0 & 0.0793 \\
0.5793 & 0.5 & 0.3293 & 0.0793 & 0.1587 \\
0.7500 & 0.6707 & 0.5 & 0.2500 & 0.3293 \\
1 & 0.9207 & 0.7500 & 0.5 & 0.5793 \\
0.9207 & 0.8413 & 0.6707 & 0.4207 & 0.5
\end{bmatrix}
\]

\[ \hat{A}_2 = (\hat{a}_{ij})_{k=5} = \]

\[
\begin{bmatrix}
0.5 & 0.2500 & 0.6707 & 0.3293 & 0.7500 \\
0.7500 & 0.5 & 0.9207 & 0.5793 & 1 \\
0.3293 & 0.0793 & 0.5 & 0.1587 & 0.5793 \\
0.6707 & 0.4207 & 0.8413 & 0.5 & 0.9207 \\
0.2500 & 0 & 0.4207 & 0.0793 & 0.5
\end{bmatrix}
\]

\[ \hat{A}_3 = (\hat{a}_{ij})_{k=5} = \]

\[
\begin{bmatrix}
0.5 & 0.7972 & 1 & 0.7440 & 0.5412 \\
0.2028 & 0.5 & 0.7028 & 0.4469 & 0.2440 \\
0 & 0.2972 & 0.5 & 0.2440 & 0.0412 \\
0.2560 & 0.5531 & 0.7560 & 0.5 & 0.2972 \\
0.4588 & 0.7560 & 0.9588 & 0.7028 & 0.5
\end{bmatrix}
\]

Step 4. Utilize Eq. (9) to fuse all the consistent complete fuzzy preference relations \( \hat{A}_k = (\hat{a}_{ij})_{k=5} \) \( (k = 1, 2, \ldots, 4) \) into a collective and complete fuzzy preference relation \( \hat{A} = (\hat{a}_{ij})_{\infty} \).

\[ \hat{A} = (\hat{a}_{ij})_{\infty} = \]

\[
\begin{bmatrix}
0.5 & 0.4177 & 0.6372 & 0.4441 & 0.4356 \\
0.5823 & 0.5 & 0.7195 & 0.5264 & 0.5180 \\
0.3628 & 0.2805 & 0.5 & 0.3069 & 0.2985 \\
0.5559 & 0.4736 & 0.6931 & 0.5 & 0.4916 \\
0.5644 & 0.4820 & 0.7015 & 0.5084 & 0.5
\end{bmatrix}
\]

Step 5. Utilize the averaging operator to fuse all the fuzzy preference degrees \( \hat{a}_{ij} \) \( (j = 1, 2, \ldots, 5) \) in the \( i \)th line of \( \hat{A} \) to obtain the average \( \hat{a}_i \) of the \( i \)th alternative over all the other alternatives.

\[ x_1 = 0.4869, \quad x_2 = 0.5692, \]
\[ x_3 = 0.3497, \quad x_4 = 0.5428, \quad x_5 = 0.5513 \]

Step 6. Rank all the alternatives \( x_i \) \( (i = 1, 2, \ldots, 5) \) and identify the optimal one(s) according to the values of \( \hat{a}_i \) \( (i = 1, 2, \ldots, 5) \).

\[ x_2 > x_3 > x_4 > x_1 > x_3 \]

Thus, the best alternative is \( x_2 \).

4. Conclusions

This study has introduced various fuzzy preference relations, including the incomplete fuzzy preference relation and the consistent incomplete fuzzy preference relation. This investigation has developed a simple and practical method, which utilizes the incomplete fuzzy preference relation involving the least judgments (that is, \( n - 1 \) judgments) to construct a consistent complete fuzzy preference relation. The most notable characteristic of the proposed method is that it requires the least judgments provided by the decision maker to create a consistent complete fuzzy preference relation, and thus it can not only reduce the time pressure faced by the decision maker but also avoid the need to check the consistency of fuzzy preference relations.

The approach developed in this study fuses individual preferences to yield collective ones and aggregates all of the information regarding each decision alternative to obtain a ranking of alternatives. Furthermore, the feasibility and effectiveness of the approach proposed here are illustrated using a numerical example.

The approach adopted in this investigation may represent a new method of solving group decision making...
problems in complex environments. Future works can further study incomplete fuzzy preferences with multi-criteria.

References


